Electric quadrupole moments of the decuplet baryons in the Skyrme model

Yongseok Oh

Department of Physics, National Taiwan University, Taipei, Taiwan 10764, Republic of China

Abstract

The electric quadrupole moments of the decuplet baryons are calculated in the bound state approach to the SU(3) Skyrme model. In this approach, all the quadrupole moments of the decuplets are found to be proportional to the third component of the baryon isospin. Contrary to the SU(3) collective model, the kaonic contribution is as important as the pionic one. The transitional quadrupole moments of hyperons are also predicted.

In the quark model, the tensor force in the inter-quark hyperfine interaction [1] leads to the D-wave admixture in the baryon wave functions, so that the ground state baryons are slightly deformed. As a result, the tensor force induces a small violation of Becchi-Morpurgo selection rule [2] that the decay $\Delta \to \mathrm{N}\gamma$ is pure $M1$ transition, and also leads to non-vanishing electric quadrupole moments of the decuplet baryons. Following the simple calculation in the oscillator model which predicts measurable Ω^- quadrupole moment [3], there are a number of publications concerning the quadrupole moments of the non-strange and strange decuplet baryons in various phenomenological models [4–13] and the results are highly model-dependent.

Recently, Leinweber *et al.* [11] studied the quadrupole moments in the lattice simulation of QCD and Butler *et al.* [12] used heavy baryon chiral perturbation theory [14] for this study. Based on the "slow rotator" approach to the SU(3) Skyrme model [15], Kroll and Schwesinger [13] investigated this problem in connection with the strangeness content of the proton. In the flavor SU(3) limit, the Skyrme model predicts maximum value of the strangeness content of the proton, $\langle s\bar{s}\rangle_p = \frac{7}{30}$, whereas $\langle s\bar{s}\rangle_p \to 0$ in the strong symmetry breaking limit. In Ref. [13], the authors claimed that the quadrupole moments are proportional to the baryon charge in the SU(3) limit and to the isospin of the baryon in the symmetry breaking limit. Their numerical results obtained with the physical kaon mass are somewhat different from those of Refs. [11,12].

In this paper, we study the electric quadrupole moments of the decuplet baryons in the bound state approach to the SU(3) Skyrme model. This approach suggested by Callan and Klebanov [16] is based on the observation that the s quark is much heavier than the light (u, d) quarks. So it treats the isospin and the strangeness in a different manner, i.e., rotational and vibrational, respectively. In this model, hyperons are described by bound states of kaon(s) in an SU(2) soliton background field. The resulting mass spectrum [16–18], electromagnetic and axial properties of hyperons [19,20] are consistent with the existing experimental data, so this model may be a reasonable starting point for understanding the hyperon structure. It is also shown [21,22] that the magnetic moments of baryons have the same structure of the quark model predictions. So, it will be interesting to investigate other electromagnetic properties of hyperons in this model. In addition, this model predicts small strangeness content of the proton, $\langle s\bar{s}\rangle_p \approx$ 3-6% [17,23]. Therefore, it will also be interesting to test the argument of Ref. [13] on the relation between the electric quadrupole moments and the SU(3) symmetry breaking.

We start with the effective action [21,24] for the Skyrme model with proper symmetry breaking terms, which reads

$$\Gamma = \int d^4x \Big\{ \frac{1}{16} F_\pi^2 \mathrm{Tr}\,(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \mathrm{Tr}\,[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2 \Big\} + \Gamma_{\mathrm{WZ}} + \Gamma_{\mathrm{SB}}, \tag{1}$$

where Γ_{WZ} is the Wess-Zumino action and Γ_{SB} includes the fact that the pion decay constant F_π ($\approx$ 186 MeV empirically) is not equal to the kaon decay constant F_K ($\approx 1.22 F_\pi$ experimentally) as well as the meson mass terms:

$$\Gamma_{\mathrm{SB}} = \int d^4x \Big\{ \frac{F_\pi^2}{48} (m_\pi^2 + 2m_K^2)\, \mathrm{Tr}\,(U + U^\dagger - 2)$$

$$
\begin{aligned}
&+\frac{\sqrt{3}}{24}F_\pi^2(m_\pi^2-m_K^2)\mathrm{Tr}\,[\lambda_8(U+U^\dagger)] \\
&+\frac{F_K^2-F_\pi^2}{48}\mathrm{Tr}\,\{(1-\sqrt{3}\lambda_8)[2m_K^2(U+U^\dagger-2) \\
&\qquad\qquad +(U\partial_\mu U^\dagger\partial^\mu U+U^\dagger\partial_\mu U\partial^\mu U^\dagger)]\}\Big\},
\end{aligned} \tag{2}
$$

with the eighth Gell-Mann matrix λ_8, where m_π and m_K are the pion and kaon mass, respectively. Taking the Callan-Klebanov ansatz [16], the SU(3)-valued chiral field U is written as

$$
U=\begin{pmatrix}\exp(i\boldsymbol{\tau}\cdot\boldsymbol{\pi}/F_\pi) & 0\\ 0 & 1\end{pmatrix}\exp\left[\frac{i2\sqrt{2}}{F_\pi}\begin{pmatrix}0 & K\\ K^\dagger & 0\end{pmatrix}\right]\begin{pmatrix}\exp(i\boldsymbol{\tau}\cdot\boldsymbol{\pi}/F_\pi) & 0\\ 0 & 1\end{pmatrix}. \tag{3}
$$

By expanding the action up to the second order of K, one can obtain the kaon-soliton effective Lagrangian. It is well-known that this effective Lagrangian with the hedgehog configuration $\exp(2i\boldsymbol{\tau}\cdot\boldsymbol{\pi}/F_\pi)=\exp[iF(r)\boldsymbol{\tau}\cdot\hat{\mathbf{r}}]$ supports a bound state solution of the lowest kaon state in the form of $K=e^{i\omega t}k(r)\boldsymbol{\tau}\cdot\hat{\mathbf{r}}\chi$, where χ is a two-component isospinor. (See Refs. [16,18] for details.)

With the given Lagrangian, we can construct the electromagnetic charge density, which reads

$$
\rho_e(\mathbf{r})=\tfrac{1}{2}Y(\mathbf{r})+I^3(\mathbf{r}), \tag{4}
$$

where

$$
\begin{aligned}
Y(\mathbf{r})&=-\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F'|k|^2-2[f(r)\omega+\lambda(r)]|k|^2,\\
I^3(\mathbf{r})&=V(r)(\delta^{bc}-\hat{\mathbf{r}}^b\hat{\mathbf{r}}^c)D^{3c}\Omega^b\\
&\quad+[A(r)\tau^b-B(r)\hat{\mathbf{r}}^b\boldsymbol{\tau}\cdot\hat{\mathbf{r}}]D^{3b}.
\end{aligned} \tag{5}
$$

(We keep the conventions of Refs. [21,24] throughout this work.) The functionals $f(r)$, $\lambda(r)$, $V(r)$, $A(r)$, and $B(r)$ are explicitly

$$
\begin{aligned}
f(r)&=1+\frac{1}{e^2F_\pi^2\chi^2}\left(F'^2+2\frac{\sin^2 F}{r^2}\right),\\
\lambda(r)&=-\frac{N_c}{2\pi^2F_\pi^2\chi^2}\frac{\sin^2 F}{r^2}F',\\
V(r)&=-2\sin^2 F\left\{\frac{F_\pi^2}{4}+\frac{1}{e^2}\left(F'^2+\frac{\sin^2 F}{r^2}\right)\right\},\\
A(r)&=\omega|k|^2f(r)\cos F-|k|^2\lambda(r)\\
&\quad+\frac{4\omega}{e^2F_\pi^2\chi^2}\{|k|^2\frac{\sin^2 F}{r^2}\cos^2\frac{F}{2}+\frac{3}{2}kk'\sin FF'\},\\
B(r)&=\omega|k|^2f(r)(1+\cos F)\\
&\quad+\frac{4\omega}{e^2F_\pi^2\chi^2}\{|k|^2\frac{\sin^2 F}{r^2}\cos^2\frac{F}{2}+\frac{3}{2}kk'\sin FF'\},
\end{aligned} \tag{6}
$$

where N_c is the number of color and $\chi = F_K/F_\pi$. In terms of the collective rotation variable $\mathcal{A}(t)$, $D^{ab} = \frac{1}{2}\mathrm{Tr}(\tau^a \mathcal{A} \tau^b \mathcal{A}^\dagger)$ and $\Omega^a = -\frac{i}{2}\mathrm{Tr}(\tau^a \mathcal{A}^\dagger \dot{\mathcal{A}})$. We take the physical pion and kaon mass, $m_\pi = 138$ MeV and $m_K = 495$ MeV, while the pion decay constant F_π and the Skyrme parameter e are chosen to be $F_\pi = 108$ MeV and $e = 4.84$ [25] to reproduce the nucleon and Δ masses. Then the energy of the bound kaon ω is calculated as 146 MeV and 209 MeV for $\chi = 1.0$ and 1.22, respectively [21].

The electric quadrupole moment operator is defined by

$$\hat{Q}_{ij} = \int \mathrm{d}^3 r \left(r_i r_j - \tfrac{1}{3} r^2 \delta_{ij} \right) \rho_e(\mathbf{r}), \tag{7}$$

and the quadrupole moment of a baryon $Q_{33}(B) = \langle B|\hat{Q}_{33}|B\rangle$ can be read from the baryon wave function [21]. From Eq. (4), it is straightforward to obtain the form of Q_{33} as

$$\begin{aligned} \hat{Q}_{33} = &-a_1 (D^{3b} R^b - 3 D^{33} R^3) \\ &- (c a_1 + a_2)(D^{3b} J_k^b - 3 D^{33} J_k^3), \end{aligned} \tag{8}$$

where

$$\begin{aligned} a_1 &= -\frac{4\pi}{45\mathcal{I}} \int \mathrm{d}r\, r^4 V(r) = \tfrac{1}{15} \langle r^2 \rangle^N_{I=1}, \\ a_2 &= \frac{4\pi}{45} \int \mathrm{d}r\, r^4 B(r). \end{aligned} \tag{9}$$

For the diagonal matrix elements one has $D^{3i} = -\frac{I^3 R^i}{I(I+1)}$, and the formula (8) can be more simplified as

$$\hat{Q}_{33}(B) = \sum_{k=1}^{2} a_k \hat{\mathcal{O}}_k \frac{I_3}{I(I+1)}, \tag{10}$$

where

$$\begin{aligned} \hat{\mathcal{O}}_1 &= I(I+1) - 3R_3^2 + c\,\hat{\mathcal{O}}_2, \\ \hat{\mathcal{O}}_2 &= \tfrac{1}{2} \left\{ J(J+1) - I(I+1) - J_k(J_k+1) \right\} - 3 J_{k,3} R_3. \end{aligned} \tag{11}$$

Here, $\mathcal{I}$ is the moment of inertia of the SU(2) soliton and $\langle r^2 \rangle^N_{I=1}$ is the isovector mean square radius of the nucleon. Therefore, for nonstrange baryons the formula (8) reproduces the SU(2) model predictions [26,27]. In Eq. (11), I (J) is the isospin (spin) of the baryon, and J_k and R are the kaon grand spin and the rotor spin, respectively [16,18]. The hyperfine constant c is obtained as $c = 0.51$ and 0.39 for $\chi = 1.0$ and 1.22, respectively.

From Eq. (5), one can see that only the isovector current contributes to the quadrupole moments since the hypercharge density is spherically symmetric. It also should be noted that the Q_{33} depends on the isospin of the baryon. As noted in Refs. [11,13], this dependence is not consistent with the quark model predictions where the quadrupole moments depend on the baryon charge. This dependence supports the argument of Ref. [13] when we consider the small strangeness content of the proton of this model. Furthermore, the Q_{33} contains

a kaonic contribution, i.e., the a_2 term, of which effect is comparable to the pionic one, whereas in the slow rotator approach of Ref. [13], it depends only on the pionic current because its moment of inertia is spherical outside the SU(2) subspace. Numerically, we have $a_1 = 0.0735$ fm^2 and $a_2 = 0.0528$ (0.0397) fm^2 for $\chi = 1.0$ (1.22).

The transitional electric quadrupole moments of the decuplet baryons can be obtained from Eq. (8). The expectation values of the operator D^{ab} can be evaluated by making use of the formula [27]

$$\begin{aligned}\langle I, I_3, R_3|D^{ab}|I', I'_3, R'_3\rangle &= \left[\frac{2I'+1}{2I+1}\right]^{1/2} (-1)^{I-I_3-I'+I'_3} \\ &\times\langle I' \; -I'_3\,; 1\,a\,|\,I\; -I_3\rangle\langle I'\,R'_3\,; 1\,b\,|\,I\,R_3\rangle. \end{aligned} \tag{12}$$

The off-diagonal matrix elements of the operators $D^{33}R^3$, $D^{33}R^b$, $D^{33}J^3_k$, and $D^{3b}J^b_k$ are given in Table I.

Our results are summarized in Tables II and III. Given in Table II are the predictions for the electric quadrupole moments of the decuplet baryons. For a comparison, the predictions of other models are also presented. The transitional electric quadrupole moments are listed in Table III. In quark models, the electric quadrupole moments are proportional to the baryon charge and the nonvanishing quadrupole moments of neutral hyperons come from the quark mass difference $m_{u,d} \neq m_s$. But this dependence should be corrected if one includes meson cloud effects as discussed in Ref. [9]. Then the quadrupole moments are nearly isospin-dependent rather than charge-dependent, which is consistent with our results. Also, our results are closer to the predictions of the heavy baryon chiral perturbation theory [12] than to those of the modified SU(3) collective model [13].

As a test of tensor force in the quark model, it has been suggested to measure the electric quadrupole moment of the Ω^- because of its long life-time. In Ref. [28], it was suggested to measure the quadrupole moment of Ω^- by studying the hyperfine splitting of the exotic Ω^--nucleus atoms if $Q_{33}(\Omega^-)$ is larger than 0.01 e$\cdot$fm^2 [6,10]. However, if the quadrupole moments have isospin dependence it may be difficult to measure the $Q_{33}(\Omega^-)$ experimentally.

As a summary, we have calculated the static and transitional electric quadrupole moments of the decuplet baryons in the bound state approach of the Skyrme model. The results show the isospin dependence of the quadrupole moments. It would hence be of great interest if such quantities could be measured experimentally, which will give very useful informations on the baryon structure.

We are grateful to Dong-Pil Min and Gwang-Ho Kim for useful discussions at the early stage of this work. Encouragements from Shin Nan Yang are also gratefully acknowledged. This research was supported in part by the National Science Council of ROC under Grant No. NSC84-2811-M002-036.

REFERENCES

[1] A. De Rújula, H. Georgi, and S. L. Glashow, Phys. Rev. D **12**, 147 (1975).
[2] C. Becchi and G. Morpurgo, Phys. Lett. **17**, 352 (1965).
[3] S. S. Gershteĭn and Yu. M. Zinov'ev, Yad. Fiz. **33**, 1442 (1981) [Sov. J. Nucl. Phys. **33**, 772 (1981)].
[4] S. S. Gershteĭn and G. V. Dzhikiya, Yad. Fiz. **34**, 1566 (1981) [Sov. J. Nucl. Phys. **34**, 870 (1981)].
[5] N. Isgur, G. Karl, and R. Koniuk, Phys. Rev. D **25**, 2394 (1982).
[6] J.-M. Richard, Z. Phys. C **12**, 369 (1982).
[7] D. Drechsel and M. M. Giannini, Phys. Lett. B **143**, 329 (1984).
[8] M. I. Krivoruchenko, Yad. Fiz. **45**, 809 (1987) [Sov. J. Nucl. Phys. **45**, 503 (1987)]; Z. Phys. C **36**, 243 (1987).
[9] W. J. Leonard and W. J. Gerace, Phys. Rev. D **41**, 924 (1990).
[10] M. I. Krivoruchenko and M. M. Giannini, Phys. Rev. D **43**, 3763 (1991).
[11] D. B. Leinweber, T. Draper, and R. M. Woloshyn, Phys. Rev. D **46**, 3067 (1992).
[12] M. N. Butler, M. J. Savage, and R. P. Springer, Phys. Rev. D **49**, 3459 (1994).
[13] J. Kroll and B. Schwesinger, Phys. Lett. B **334**, 287 (1994).
[14] E. Jenkins and A. V. Manohar, Phys. Lett. B **255**, 558 (1991).
[15] B. Schwesinger and H. Weigel, Nucl. Phys. **A450**, 461 (1992).
[16] C. G. Callan and I. Klebanov, Nucl. Phys. **B262**, 365 (1985); C. G. Callan, K. Hornbostel, and I. Klebanov, Phys. Lett. B **202**, 269 (1988).
[17] J.-P. Blaizot, M. Rho, and N. N. Scoccola, Phys. Lett. B **209**, 27 (1988).
[18] N. N. Scoccola, D.-P. Min, H. Nadeau, and M. Rho, Nucl. Phys. **A505**, 497 (1989).
[19] J. Kunz and P. J. Mulders, Phys. Lett. B **231**, 335 (1989); Phys. Rev. D **41**, 1578 (1990); E. M. Nyman and D. O. Riska, Nucl. Phys. **B325**, 593 (1989); D.-P. Min, Y. S. Koh, Y. Oh, and H. K. Lee, Nucl. Phys. **A530**, 698 (1991).
[20] Y. Kondo, S. Saito, and T. Otofuji, Phys. Lett. B **236**, 1 (1990); **256**, 316 (1991); Y. Kondo and S. Saito, Few-Body Syst. **12**, 113 (1992); K. Dannbom and D. O. Riska, Nucl. Phys. **A548**, 669 (1992).
[21] Y. Oh, D.-P. Min, M. Rho, and N. N. Scoccola, Nucl. Phys. **A534**, 493 (1991).
[22] Y. Oh and B.-Y. Park (in preparation).
[23] C. W. Wong, D. Vuong, and K.-C. Chu, Nucl. Phys. **A515**, 686 (1990).
[24] D. O. Riska and N. N. Scoccola, Phys. Lett. B **265**, 188 (1991).
[25] G. S. Adkins and C. R. Nappi, Nucl. Phys. **B233**, 109 (1984).
[26] A. Wirzba and W. Weise, Phys. Lett. B **188**, 6 (1987).
[27] G. S. Adkins, in *Chiral Solitons*, edited by K.-F. Liu (World Scientific, Singapore, 1987).
[28] R. M. Sternheimer and M. Goldhaber, Phys. Rev. A **8**, 2207 (1973).

TABLES

$\langle p, J_z=\frac{1}{2}\|D^{33}R^3\|\Delta^+, J_z=\frac{1}{2}\rangle = -\frac{\sqrt{2}}{6}$	$\langle p, J_z=\frac{1}{2}\|D^{3b}R^b\|\Delta^+, J_z=\frac{1}{2}\rangle = 0$
$\langle \Lambda, J_z=\frac{1}{2}\|D^{33}R^3\|\Sigma^0, J_z=\frac{1}{2}\rangle = 0$	$\langle \Lambda, J_z=\frac{1}{2}\|D^{3b}R^b\|\Sigma^0, J_z=\frac{1}{2}\rangle = 0$
$\langle \Lambda, J_z=\frac{1}{2}\|D^{33}R^3\|\Sigma^{*,0}, J_z=\frac{1}{2}\rangle = 0$	$\langle \Lambda, J_z=\frac{1}{2}\|D^{3b}R^b\|\Sigma^{*,0}, J_z=\frac{1}{2}\rangle = 0$
$\langle \Sigma^a, J_z=\frac{1}{2}\|D^{33}R^3\|\Sigma^{*,a}, J_z=\frac{1}{2}\rangle = -\frac{\sqrt{2}}{6}a$	$\langle \Sigma^a, J_z=\frac{1}{2}\|D^{3b}R^b\|\Sigma^{*,a}, J_z=\frac{1}{2}\rangle = -\frac{\sqrt{2}}{3}a$
$\langle \Xi^a, J_z=\frac{1}{2}\|D^{33}J^3_k\|\Xi^{*,a}, J_z=\frac{1}{2}\rangle = 0$	$\langle \Xi^a, J_z=\frac{1}{2}\|D^{3b}J^b_k\|\Xi^{*,a}, J_z=\frac{1}{2}\rangle = 0$
$\langle p, J_z=\frac{1}{2}\|D^{33}J^3_k\|\Delta^+, J_z=\frac{1}{2}\rangle = 0$	$\langle p, J_z=\frac{1}{2}\|D^{3b}J^b_k\|\Delta^+, J_z=\frac{1}{2}\rangle = 0$
$\langle \Lambda, J_z=\frac{1}{2}\|D^{33}J^3_k\|\Sigma^0, J_z=\frac{1}{2}\rangle = \frac{1}{6}$	$\langle \Lambda, J_z=\frac{1}{2}\|D^{3b}J^b_k\|\Sigma^0, J_z=\frac{1}{2}\rangle = \frac{1}{2}$
$\langle \Lambda, J_z=\frac{1}{2}\|D^{33}J^3_k\|\Sigma^{*,0}, J_z=\frac{1}{2}\rangle = -\frac{\sqrt{2}}{6}$	$\langle \Lambda, J_z=\frac{1}{2}\|D^{3b}J^b_k\|\Sigma^{*,0}, J_z=\frac{1}{2}\rangle = 0$
$\langle \Sigma^a, J_z=\frac{1}{2}\|D^{33}J^3_k\|\Sigma^{*,a}, J_z=\frac{1}{2}\rangle = \frac{\sqrt{2}}{12}a$	$\langle \Sigma^a, J_z=\frac{1}{2}\|D^{3b}J^b_k\|\Sigma^{*,a}, J_z=\frac{1}{2}\rangle = \frac{\sqrt{2}}{6}a$
$\langle \Xi^a, J_z=\frac{1}{2}\|D^{33}J^3_k\|\Xi^{*,a}, J_z=\frac{1}{2}\rangle = \frac{2\sqrt{2}}{9}(a+\frac{1}{2})$	$\langle \Xi^a, J_z=\frac{1}{2}\|D^{3b}J^b_k\|\Xi^{*,a}, J_z=\frac{1}{2}\rangle = \frac{2(\sqrt{2}-1)}{9}(a+\frac{1}{2})$

TABLE I. Off-diagonal matrix elements of the operators in Eq. (8).

Particle	Δ^{++}	Δ^+	Δ^0	Δ^-	Σ^{*+}	Σ^{*0}	Σ^{*-}	Ξ^{*0}	Ξ^{*-}	Ω^-
Ref. [3]										1.8
Ref. [4]	−6.6	−3.3	0.0	3.3						
Ref. [5]	−9.8	−4.9	0.0	4.9						3.1
Ref. [6]										0.4
Ref. [7]	−17.8	−8.9	0.0	8.9						
Ref. [8]	−12.6	−6.3	0.0	6.3						
Ref. [9]	−6.0	−2.1	1.8	5.7	−2.2	−0.01	2.0	−0.6	1.0	0.6
Ref. [10]	−9.3	−4.6	0.0	4.6	−5.4	−0.7	4.0	−1.3	3.4	2.8
Ref. [11]	−2.7	−1.3	0.0	1.3	0.2	0.5	1.0	0.5	0.8	0.5
Ref. [12]	−8.0	−3.0	1.2	6.0	−7.0	−1.3	4.0	−3.5	2.0	0.9
Ref. [13]	−8.7	−3.1	2.4	8.0	−4.2	0.5	5.2	−0.7	3.5	2.4
This Work (I)	−8.8	−2.9	2.9	8.8	−8.2	0.0	8.2	−6.0	6.0	0.0
This Work (II)	−8.8	−2.9	2.9	8.8	−7.1	0.0	7.1	−4.6	4.6	0.0

TABLE II. Electric quadrupole moments of the decuplet baryons in the unit of 10^{-2} e·fm^2. The work (I) and (II) correspond to $\chi = 1.0$ and 1.22, respectively.

$\Delta^+ \to p$	$\Delta^0 \to n$	$\Sigma^{*,0} \to \Lambda$	$\Sigma^{*,+} \to \Sigma^+$	$\Sigma^{*,0} \to \Sigma^0$	$\Sigma^{*,-} \to \Sigma^-$	$\Xi^{*,0} \to \Xi^0$	$\Xi^{*,-} \to \Xi^-$
−5.20	5.20	−4.83	−0.93	0	0.93	2.91	−2.91

TABLE III. Transitional electric quadrupole moments of the decuplet baryons with $\chi = 1.22$ in the unit of 10^{-2} e·fm^2.